Fuzzy Sets and Systems

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If my boss ever finds out I’m running his plant with fuzzy control, I’m in real trouble!

Unidentified Plant Engineer,
Early 1980s

“I can’t let the man continue with fuzzy math.”

Governor George W. Bush

First Bush/Gore Presidential Debate

October 3, 2000

What does "fuzzy" mean?

Fuzzy sets and fuzzy logic were introduced by Lotfi A. Zadeh in 1965. Zadeh was almost single-handedly responsible for the early development in this field.

Seminal References

1. L.A. Zadeh, “Fuzzy sets”, Inf. Control 8, 338-353, 1965.
2. L.A. Zadeh, “Fuzzy sets as a basis for a theory of possibility”, Fuzzy Sets and Systems 1, 3-28, 1978.

What is a fuzzy set?

A fuzzy set is a collection of objects with graded membership.

Graded Membership?

Two Examples of "Sets"

1. All employees of XYZ who are over 1.8 m in height.
2. All employees of XYZ who are tall.

The first example is a classical set -- we have a universe (all XYZ employees) and a membership rule that divides the universe into members (those over 1.8 m) and nonmembers.

The second example is a fuzzy set -- some employees are definitely in the set and some are definitely not in the set, but some are “borderline”.

This distinctions between the “ins”, the “outs”, and the “borderlines” is made more exact by the membership function, $\mu_A(x)$.

$\mu_A(x)$

If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), what would the function $\mu_A(x)$ look like?

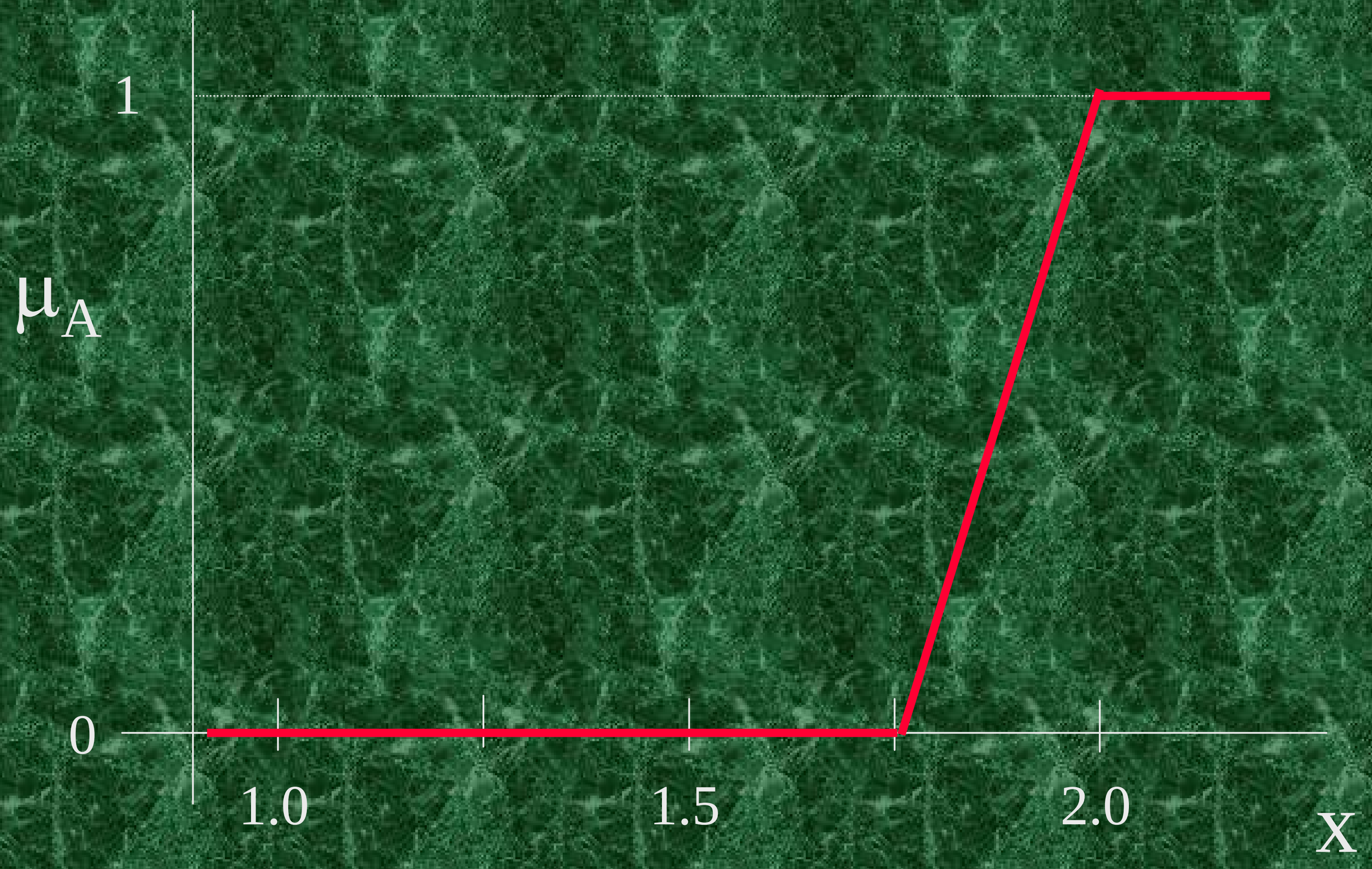
$\mu_A(x)$, continued

$\mu_A(x) = 1$ if x is definitely tall

$\mu_A(x) = 0$ if x is definitely <u>not</u> tall

$0 < \mu_A(x) < 1$ for borderline cases

A possible form for $\mu_A(x)$:



i.e. anyone over 2.0 m is definitely tall,

anyone under 1.75 m is definitely not tall

anyone between 1.75 m and 2.0 m is partly tall and partly not tall

More on fuzzy sets

1. The support of A:

   $$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

For our example, $\text{supp}(A) = (1.75, \infty)$

2. The crossover point(s) of A are

   $$\{x \in X \mid \mu_A(x) = 0.5\}$$

For our example, there is only one crossover point at $x = 1.875$

More on fuzzy sets, continued

3. The height of A:

$$hgt(A) = \sup_{x \in X} \mu_A(x)$$

In our example, hgt(A) = 1

4. A is normalized if hgt(A) = 1. In our example, A is normalized.

More on fuzzy sets, continued

5. The union of two fuzzy sets A and B (both contained in X):

   $$\mu_{A \cup B}(x) = \max(\mu_A(x)\ ,\ \mu_B(x)\ )$$

6. The intersection of two fuzzy sets A and B (both contained in X):

   $$\mu_{A \cap B}(x) = \min(\mu_A(x)\ ,\ \mu_B(x)\ )$$

Example 3: Intersection and Union of Fuzzy Sets

Individual	Height (m)	Handedness
1	1.6	ambidextrous
2	1.9	right-handed
3	2.1	right-handed
4	1.7	left-handed

Example 3: Intersection and Union of Fuzzy Sets, cont'd

Here

X = set of all four individuals

A = all members of X who are tall

B = all members of X who are left-handed

Example 3: Intersection and Union of Fuzzy Sets, cont'd, 2

Individual	$\mu_A(x)$	$\mu_B(x)$
1	0	0.5
2	0.6	0
3	1	0
4	0	1

Note: both A and B are normalized

Example 3: Intersection and Union of Fuzzy Sets, cont'd, 3

$A \cup B$ is the fuzzy set of X members that are tall <u>or</u> left-handed

Individual	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A \cup B}(x)$
1	0	0.5	0.5
2	0.6	0	0.6
3	1	0	1
4	0	1	1

Example 3: Intersection and Union of Fuzzy Sets, cont'd, 4

Therefore, all four individuals have some degree of membership in $A \cup B$, and two are definitely inside $A \cup B$

$A \cap B$ is the fuzzy set of X members that are tall <u>and</u> left-handed.

Example 3: Intersection and Union of Fuzzy Sets, cont'd, 5

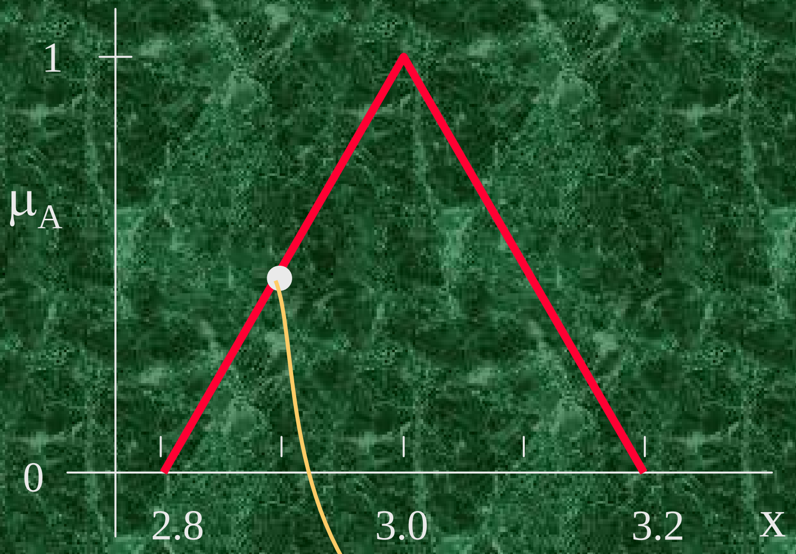
Individual	μ_A	μ_B	$\mu_{A \cap B}$
1	0	0.5	0
2	0.6	0	0
3	1	0	0
4	0	1	0

Therefore, the intersection of A and B is empty.

Fuzzy Functions

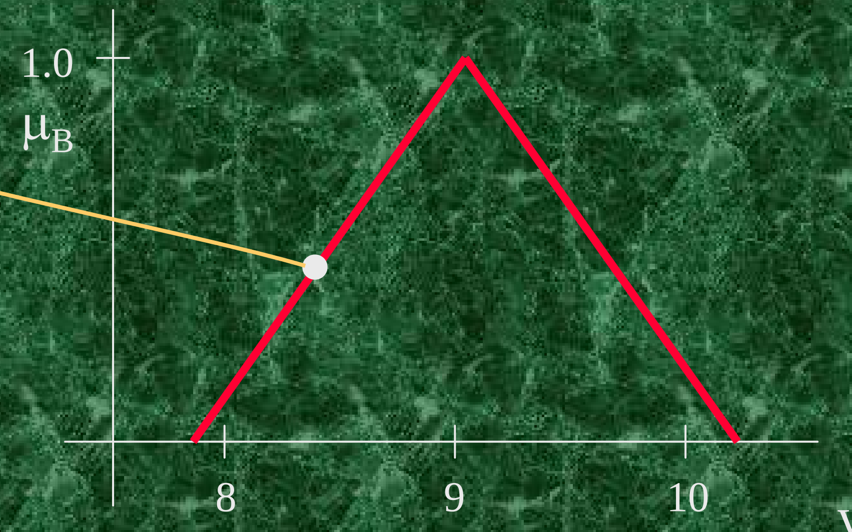
- 1. Ordinary function operating on the elements of a fuzzy set:

  e.g. the function $f(x) = x^2$ maps the fuzzy set A (numbers around 3) to the fuzzy set B (numbers around 9).



$f : A \rightarrow B$

$f(x) = x^2$

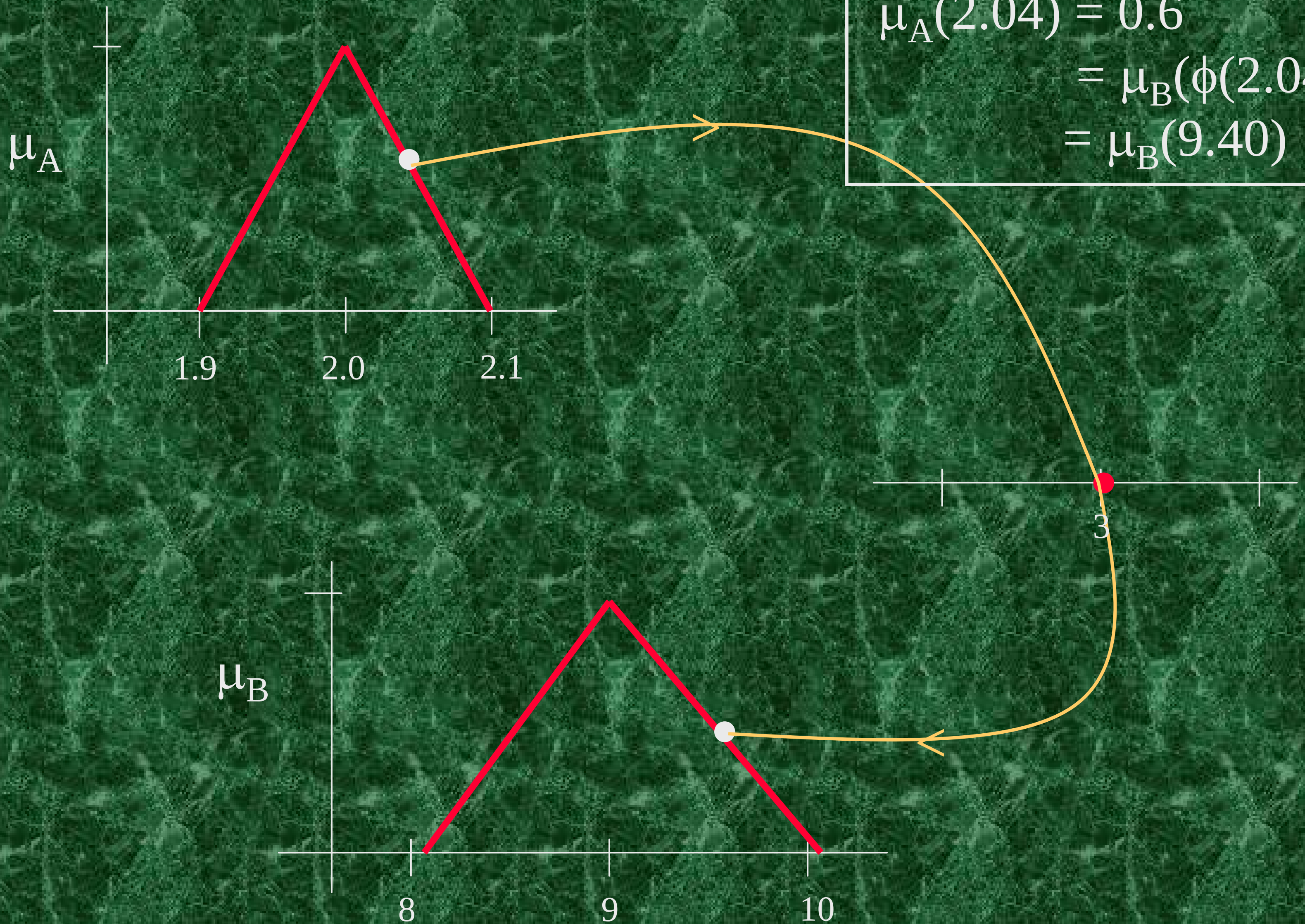


$$
\begin{aligned}
\mu_A(2.9) &= 0.5 \\
&= \mu_B(f(2.9)) \\
&= \mu_B(8.4)
\end{aligned}
$$

Fuzzy Functions, continued

- 2. Fuzzifying function, operating on elements of a classical set.

  e.g. if $\phi(x) = x^a$ (where a is a fuzzy parameter near 2) operates on a single value of x (say x = 3), the image is a fuzzy set B (numbers around 9). $\phi(x)$ can also be viewed as a fuzzy set of functions.



$$\mu_A(2.04) = 0.6$$
$$= \mu_B(\phi(2.04))$$
$$= \mu_B(9.40)$$

Fuzzy Arithmetic

It is possible to extend the ordinary binary operations of arithmetic (addition, subtraction, multiplication and division) to fuzzy sets, as long as these operations are defined for the elements of the fuzzy sets.

We define $A + B$ and $A * B$ as follows:

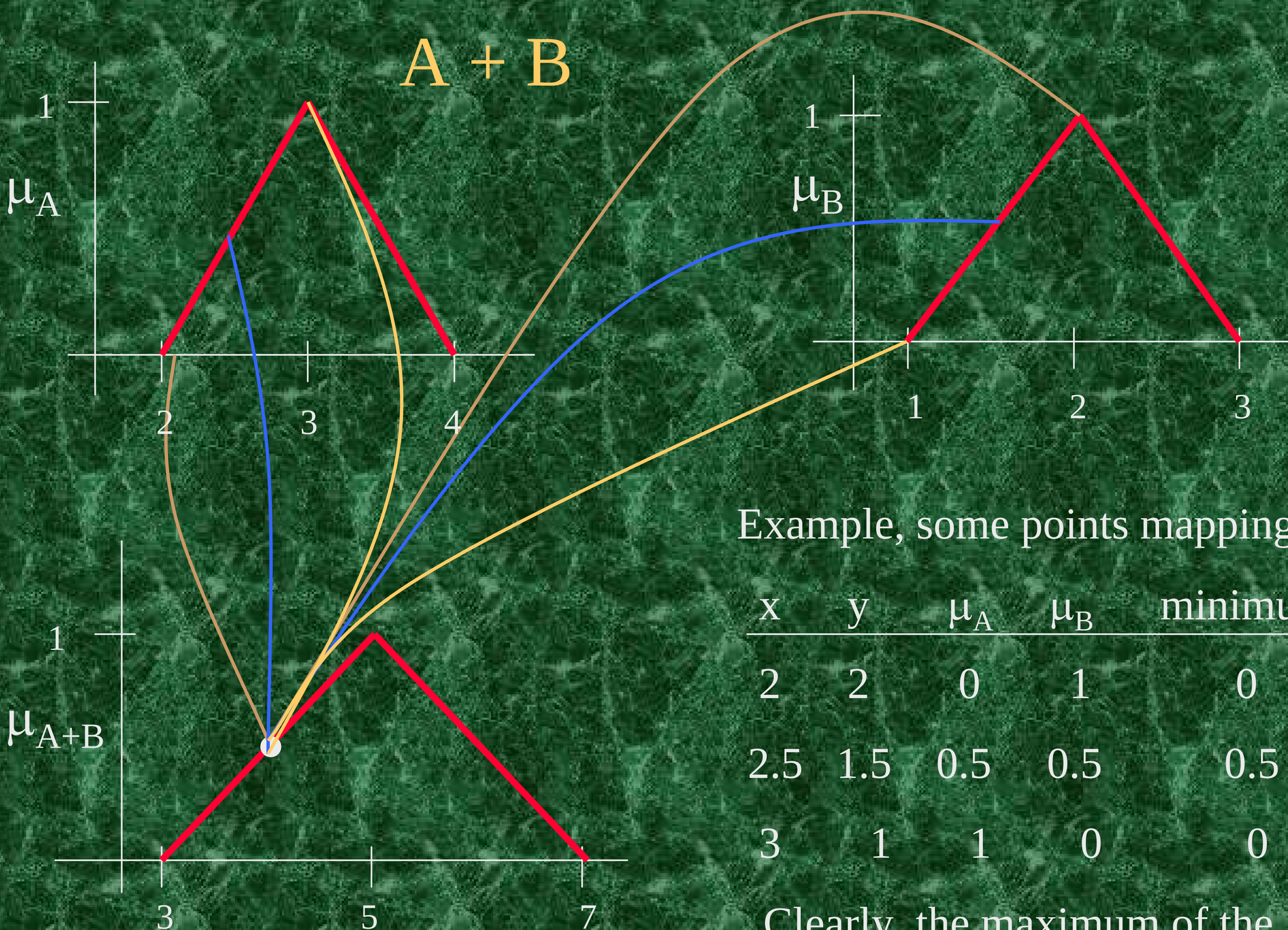
A + B is the set of all possible sums x + y with x from A and y from B. The membership function for an element z of A + B is the maximum (over all (x,y) pairs that give x + y = z) of the minima of the membership functions of x and y.

A $*$ B is the set of all possible products x $*$ y with x from A and y from B. The membership function for an element z of A $*$ B is the maximum (over all (x,y) pairs that give x $*$ y = z) of the minima of the membership functions of x and y.

These definitions should be much clearer with some examples.

Examples for A (a fuzzy set of numbers near 3) and B (a fuzzy set of numbers near 2) follow.

A + B

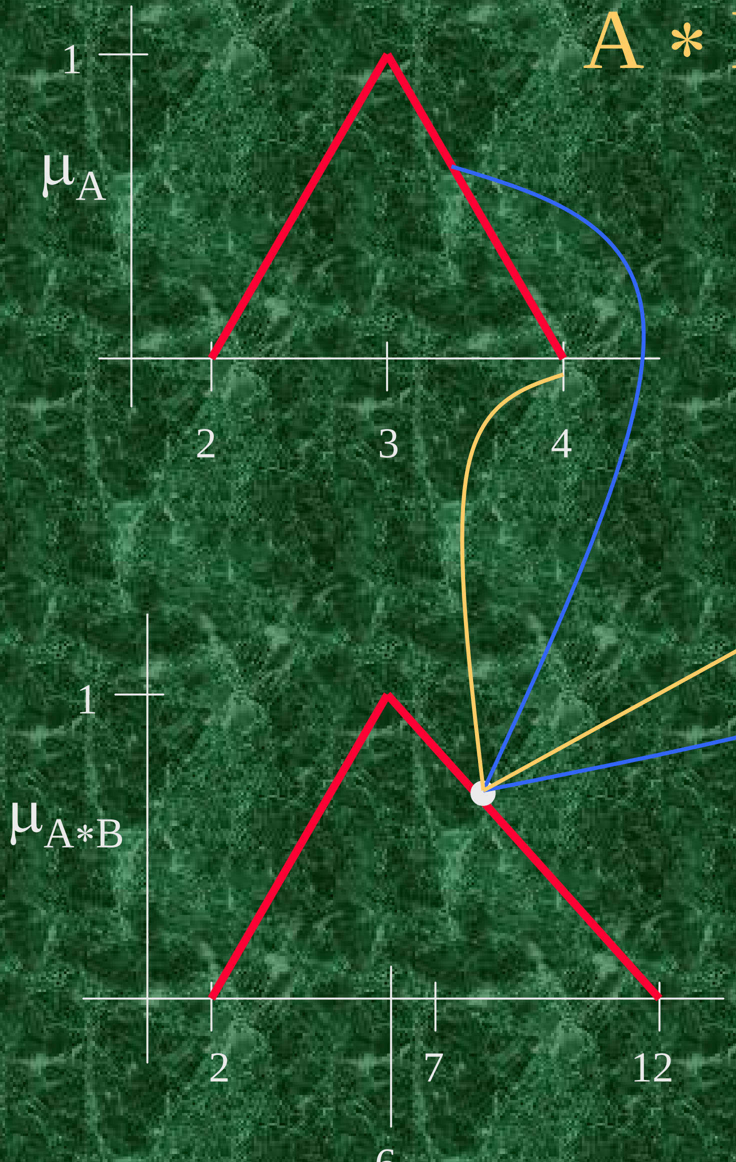


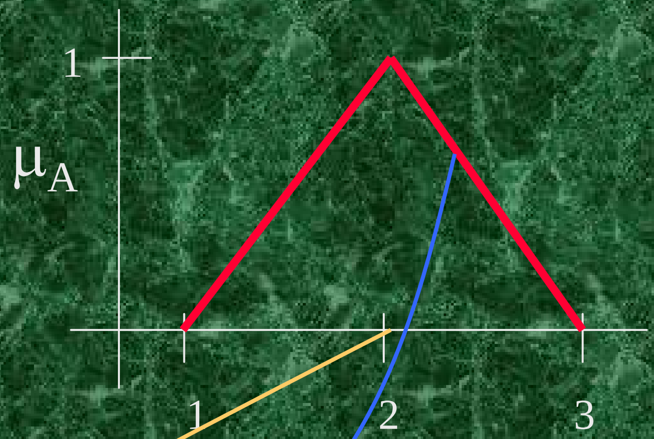
Example, some points mapping onto 4

x	y	μ_A	μ_B	minimum
2	2	0	1	0
2.5	1.5	0.5	0.5	0.5
3	1	1	0	0

Clearly, the maximum of the minima is 0.5, which is plotted as μ.

$A * B$





Example, some points mapping onto 8

x	y	μ_A	μ_B	minimum
4	2	0	1	0
3.2	2.5	0.8	0.5	0.5
3.37	2.37	0.63	0.63	0.63

Fuzzy Arithmetic, continued

Subtraction and division are defined as the obvious extensions to addition and multiplication, respectively. Of course, division will not be defined if zero is an element with non-zero membership function of the fuzzy set that is being used as the divisor.

Some “Fuzzy” Applications

Fuzzy Sets and Significant Figures

One straightforward application of fuzzy sets is the re-examination of the idea of "significant figures". From a fuzzy viewpoint,

x = 3.5 means x is an element of a fuzzy set whose membership function is 1 between 3.45 and 3.55 and 0 elsewhere.

y = 3.49 means y is an element of another fuzzy set whose membership function is 1 between 3.485 and 3.495 and 0 elsewhere.

Significant Figures, continued

To see how this formulation would be useful, consider the equation

$$z = \exp(-(x+y))$$

where $x = 1.00$ and $y = 2.5$. The membership functions for x and y are then given by

$$\mu_x = 1 \text{ for } x \in [0.995, 1.005],$$
$$= 0 \text{ elsewhere} \qquad \text{and}$$

$$\mu_y = 1 \text{ for } y \in [2.45, 2.55],$$
$$= 0 \text{ elsewhere.}$$

Significant Figures, continued (2)

Then,

$$\mu_{x+y} = 1 \text{ for } x \in [3.440, 3.555], \quad = 0 \text{ elsewhere} \qquad \text{and}$$

$$\mu_z = 1 \text{ for } x \in [0.02858, 0.03206], \quad = 0 \text{ elsewhere}$$

Returning to the normal world of significant figures, we would say that z has between 1 and 2 significant figures (i.e. the precision is somewhere between z = 0.03 and z = 0.030).

Significant Figures, continued (3)

Note that because the original membership function values are either 0 or 1, the evaluation of the maximum of a series of minima for the resultant membership function is much easier.

Fuzzy Control

State Space Representations in a Fuzzy World

It is possible to consider systems such as the FDDLDS*

$$x_{t+1} = A\,x_t + B\,u_t$$

$$y_{t+1} = C\,x_{t+1}$$

and its continuous counterpart, the FDSLDS**

$$\frac{dx}{dt} = A\,x + B\,u$$

$$y = C\,x$$

* Finite dimensional discrete linear dynamical system

** Finite dimensional smooth linear dynamical system with x and u representing fuzzy vectors of states and controls and show that these representations are possible. In addition, a fuzzy Kalman filter for each system can be derived under a fairly modest set of assumptions. However, control systems in the fuzzy world are often much less mathematically sophisticated.

Early Fuzzy Control Systems

The first system controlled by fuzzy logic was a small steam engine; the algorithm was due to Mamdani and Assilian at the University of London in 1974*.

E.H. Mamdani

* E. H. Mamdani, "Application of fuzzy algorithms for control of simple dynamic plant", Proc. Inst. Elec. Eng., 121 1585-1588 (1974).

Early Fuzzy Control Systems, continued

A much larger scale system was the first commercial application of fuzzy control -- Holmblad and Østergaard were able to use a fuzzy control scheme to run a cement kiln** in Denmark in the 1970s.

** P. Holmblad and J.-J. Østergaard, “Control of a Cement Kiln by Fuzzy Logic”, pp. 398-399 in M. M. Gupta and E. Sanchez, eds. “Fuzzy Information and Decision Processes, North-Holland, Amsterdam (1982).

Early Fuzzy Control Systems, continued

Both of these control schemes were significantly better than conventional automatic control systems. In addition, neither required a detailed mathematical model of the process, relying instead on simple “rules-of-thumb” for when and how the process should be adjusted.

Let us examine two very simple fuzzy control systems.

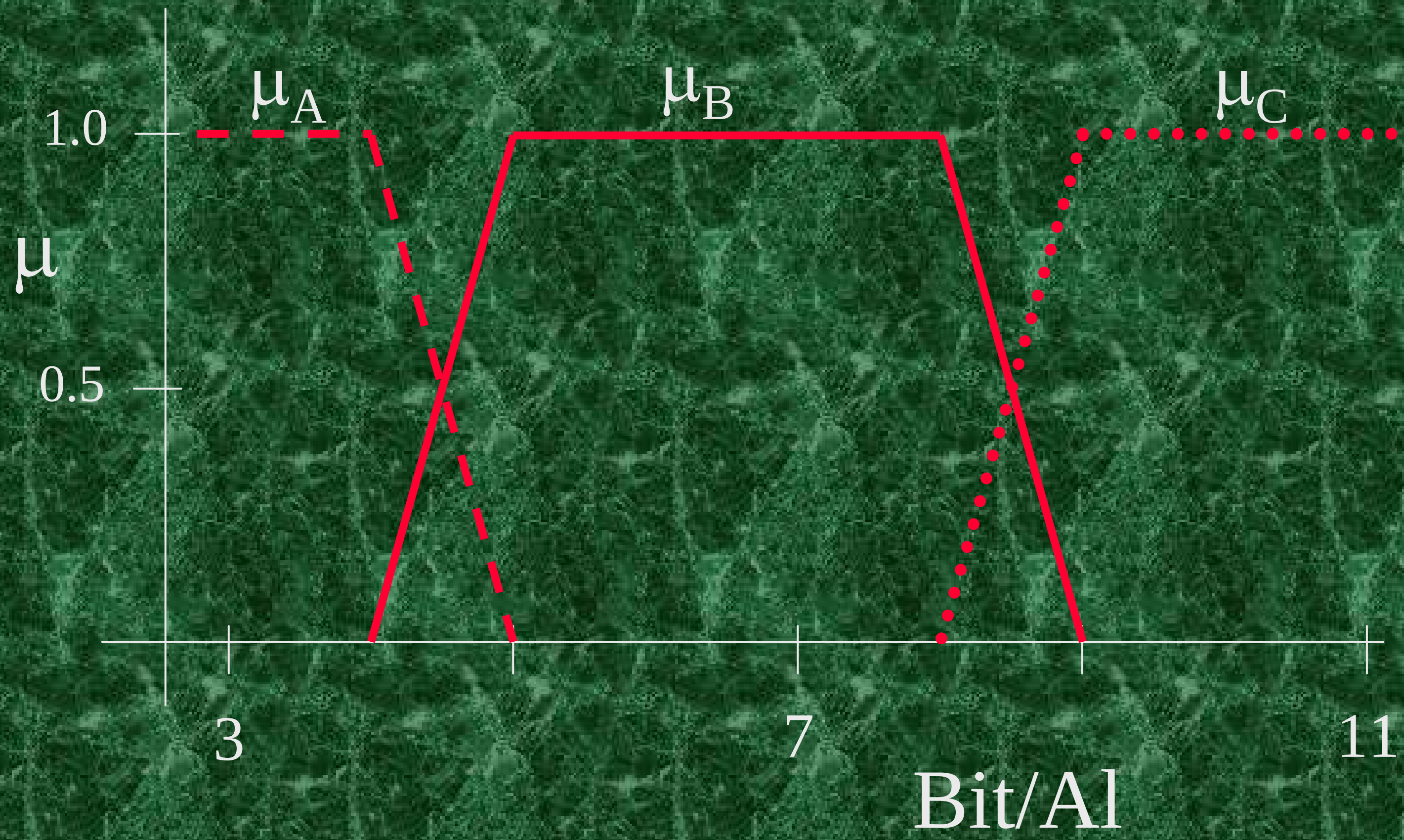
Simple Example: Optimal Water Addition in Oil Sand Extraction

One of the control variables in the slurrying step of the extraction process for producing bitumen from oil sand is the amount of water added. Batch tests indicate that if bitumen/aluminum in the oil sand is in the range of 5 to 8, “normal” amounts of slurry water are best. Higher Bit/Al assays suggest “less than normal” amounts of slurry water would be optimal; Bit/Al assays below 5 suggest “more than normal” slurry water be used.

Optimal Water Addition in Oil Sand Extraction, continued

Let A be the fuzzy set of "low" Bit/Al values, B be the fuzzy set of "normal" Bit/Al values and C be the fuzzy set of "high" Bit/Al values. We can also interpret membership in B as calling for "normal" amounts of slurry water, membership in A as calling for "high" amounts of slurry water, and membership in C as calling for "low" amounts of slurry water.

Membership Functions for A, B and C



Control Rules

Bit/Al	**Slurry Water**
low (A)	high
normal (B)	normal
high (C)	low

Ties (situations in which there is equal membership function values in two fuzzy sets) go to “normal”.

Applying the Control Rules

For example, if Bit/Al = 4.25, reference to the membership function plot gives $\mu_A = 0.75$, $\mu_B = 0.25$ and $\mu_C = 0$. Thus the oil sand is best described as belonging to A and should have higher than normal slurry water.

Another oil sand with Bit/Al = 8.5 gives $\mu_A = 0$, $\mu_B = 0.5$ and $\mu_C = 0.5$. Invoking the tie-breaking rule suggests that normal amounts of slurry water be used.

Advantages of Fuzzy Control Scheme

The advantages of this fuzzy control scheme are that it is simple and “scale free” (the operator of the large scale process is free to set his own “normal”, “lower” and “higher” values for slurry water).

The same result could have been obtained without using fuzzy sets, but fuzzy sets give a more natural formulation.

Some tuning of the control scheme is possible by adjusting the membership functions or by introducing more fuzzy sets to span the range in Bit/Al values.

A Control Scheme Based on Two Observations

A car is being driven through the mountains. How do we operate the gas pedal to maintain speeds at around 80 km/h (the speed limit)?

We will observe both the instantaneous speed and whether the car is accelerating, decelerating or keeping a constant speed.

Control Rules

Speed	Action
high, rising	decelerate
high, constant	decelerate
high, dropping	no action
medium,rising	decelerate
medium, constant	no action
medium, dropping	accelerate
low, rising	no action
low, constant	accelerate
low, dropping	accelerate

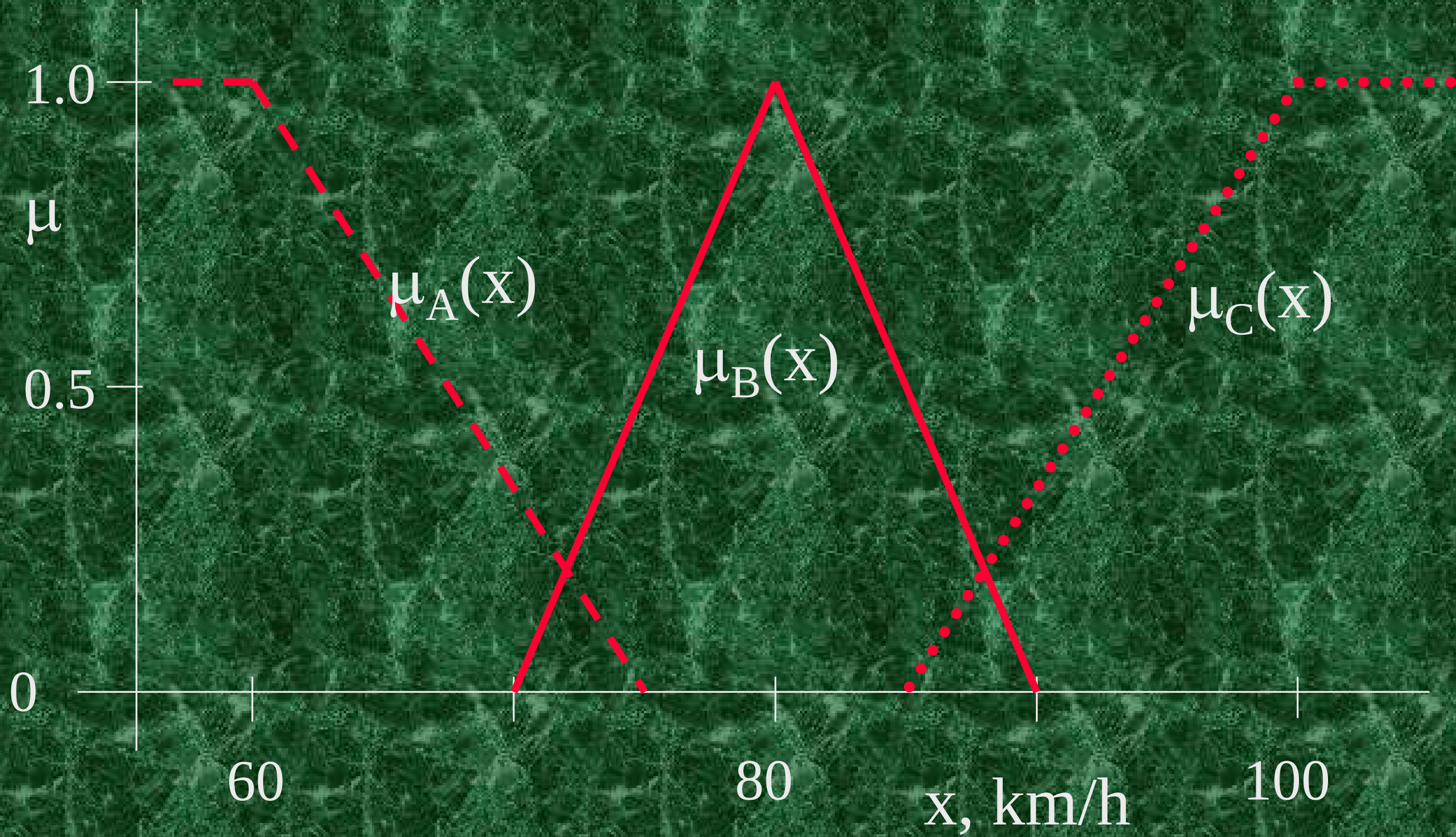
Let

A = fuzzy set of low speeds

B = fuzzy set of medium speeds

C = fuzzy set of high speeds

Membership Functions for Speeds



Further, let

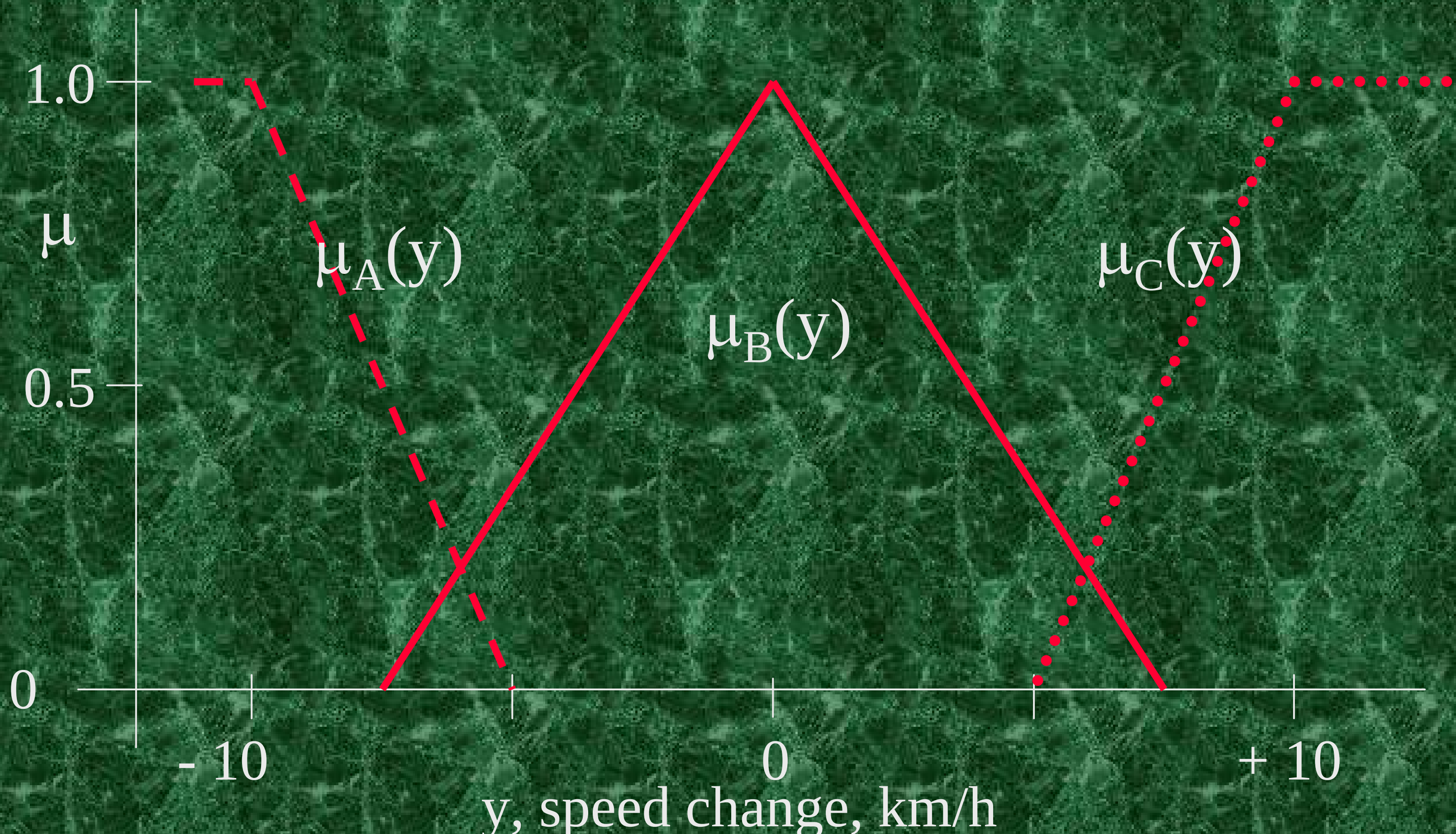
D = fuzzy set of speed losses

E = fuzzy set of constant speeds

F = fuzzy set of speed gains

Note: these speed changes are all over the interval between successive observations

Membership Functions for Speed Changes



How to Use the Control Chart and the Membership Functions

For a given (x, y) pair, find $\mu_*(x, y)$ for

$$
* = \begin{array}{ccc} A \cap D & A \cap E & A \cap F \\ B \cap D & B \cap E & B \cap F \\ C \cap D & C \cap E & C \cap F \end{array}
$$

Determine which of these intersections gives the highest value for μ, then implement the decision indicated by the intersection,

Example

For example, let (x, y) = (90, -5). Then $\mu_A(x) = 0$, $\mu_B(x) = 0$, $\mu_C(x) = 0.33$, $\mu_D(y) = 0$, $\mu_E(y) = 0.33$, and $\mu_F(y) = 0$.

Using the rules for intersection described earlier,

$$\mu_{A \cap D} = 0 \qquad \mu_{A \cap E} = 0 \qquad \mu_{A \cap F} = 0$$

$$\mu_{B \cap D} = 0 \qquad \mu_{B \cap E} = 0 \qquad \mu_{A \cap F} = 0$$

$$\mu_{C \cap D} = 0 \qquad \mu_{C \cap E} = 0.33 \qquad \mu_{C \cap F} = 0$$

Therefore,

$$\max_{*} \mu_*(x, y) = \mu_{C \cap E}$$
$$= 0.33$$

and the speed is best described as high and constant. Referring back to the control chart, we should decelerate.

Notes:

1. We will need a rule for breaking ties -- the best option is to have a tie result in the normal (i.e. no action) state.

2. Note that the membership functions were constructed to overlap -- this is a necessity (Discussion: why is this necessary?).

3. Some tuning of the control system is possible by changing the membership function. If needed, the system could be re-constructed using more fuzzy sets to span the possible speeds and speed changes.

4. The control chart is obviously just a collection of “rules-of-thumb” from an experienced operator. Changing these into membership functions and control actions is the essence of a fuzzy control scheme.

Translating English into Membership Functions

In both of the control examples, once we had membership functions for the variables of interest the process became almost automatic. Is there any general guidance on translating English into membership functions?

Dubois and Prade* give the straightforward example of the set

* Didier Dubois and Henri Prade, "Fuzzy Sets and Systems: Theory and Applications", Academic, New York, 1980. -- page 257.

Translating English into Membership Functions, 2

{true, more or less true, borderline, more or less false, false} mapping onto the set of membership function values { 1, 0.75, 0.5, 0.25, 0 }.

Didier Dubois

Henri Prade

Translating English into Membership Functions, 3

A more detailed mapping between English and membership functions is possible using, for example, the studies of Simpson in 1944* and Hakel in 1968**.

Milton D. Hakel

*Ray Simpson, "The Specific Meanings of Certain Terms Indicating Differing Degrees of Frequency", The Quarterly Journal of Speech, 30, 1944, pp 328-330.

** Milton D. Hakel, "How Often is Often?", American Psychologist, 23, 1968, pp 533-534.

Ray Simpson asked 355 high school and college students to place 20 frequency terms like “often” on a scale between 0 and 100. How many times out of 100 was “often”? Milton Hakel repeated the experiment in 1968. A sampling of the results from the two studies follows:

Word	1944 Median Score	1968 Median Score
always	99	100
very often	88	87
usually	85	79
often	78	74
…	…	…
sometimes	20	29
occasionally	20	28
not often	13	16
usually not	10	16
seldom	10	9
rarely	5	5
never	0	0

Using either of these scales (or their average median scores) and dividing by 100 gives a mapping between these subjective terms and membership in the fuzzy set “ALWAYS”.

Similar surveys can put other descriptive terms on a quantitative scale. For instance, we are interested in learning how many trials would be successful (out of 100) if the likelihood of success was described by the words

Probable

Very probable

Likely

Certain

Improbable

Possible

Impossible

Expert Systems

Expert Systems

Many tasks in industry are performed by highly skilled workers with years of experience. Some of these jobs are difficult to describe with equations, but the worker uses rules-of-thumb and his experience to complete the task. An **expert system** tries to capture the worker's knowledge in a way that can be programmed as an automatic control scheme on a computer. The most natural way to do this is to use the skilled operator's "rules-of-thumb" to write fuzzy control algorithms.

Expert Systems, continued

If we are constructing a fuzzy control scheme from the verbal instructions of an experienced operator, we will need to turn his English into membership functions. It is better to survey to learn how to quantify his English after we know what terms are most crucial to quantify. Our example of adjusting the water to optimize oil sand extraction is a very simple example of an expert system

Possibility Theory

Probability vs Possibility

Zadeh* observed that, if the membership function denoted the possible occurrence of an event or outcome, then the membership function could be viewed as a generalization of classical probability. This generalization was termed "possibility" by Zadeh.

L.A. Zadeh, "Fuzzy sets as a basis for a theory of possibility", Fuzzy Sets and Systems 1, 3-28, 1978.

Bart Kosko was able to show* that classical probability theory is a special case of “fuzziness”. Kosko used a slightly different formulation for fuzzy sets in which the membership function is replaced by the extent to which one set can be considered a subset of another set.

* Bart Kosko, “Neural Networks and Fuzzy Systems”, Prentice Hall, Englewood Cliffs, NJ, 1991.

Possibility Example -- POMEL

A straightforward example in which we use the membership function in a manner similar to a cumulative probability distribution is POMEL analysis (Possibility Model for Environmental Liability)*,**. This model is used for estimating the liability for environmental damage from a portfolio of many different sites.

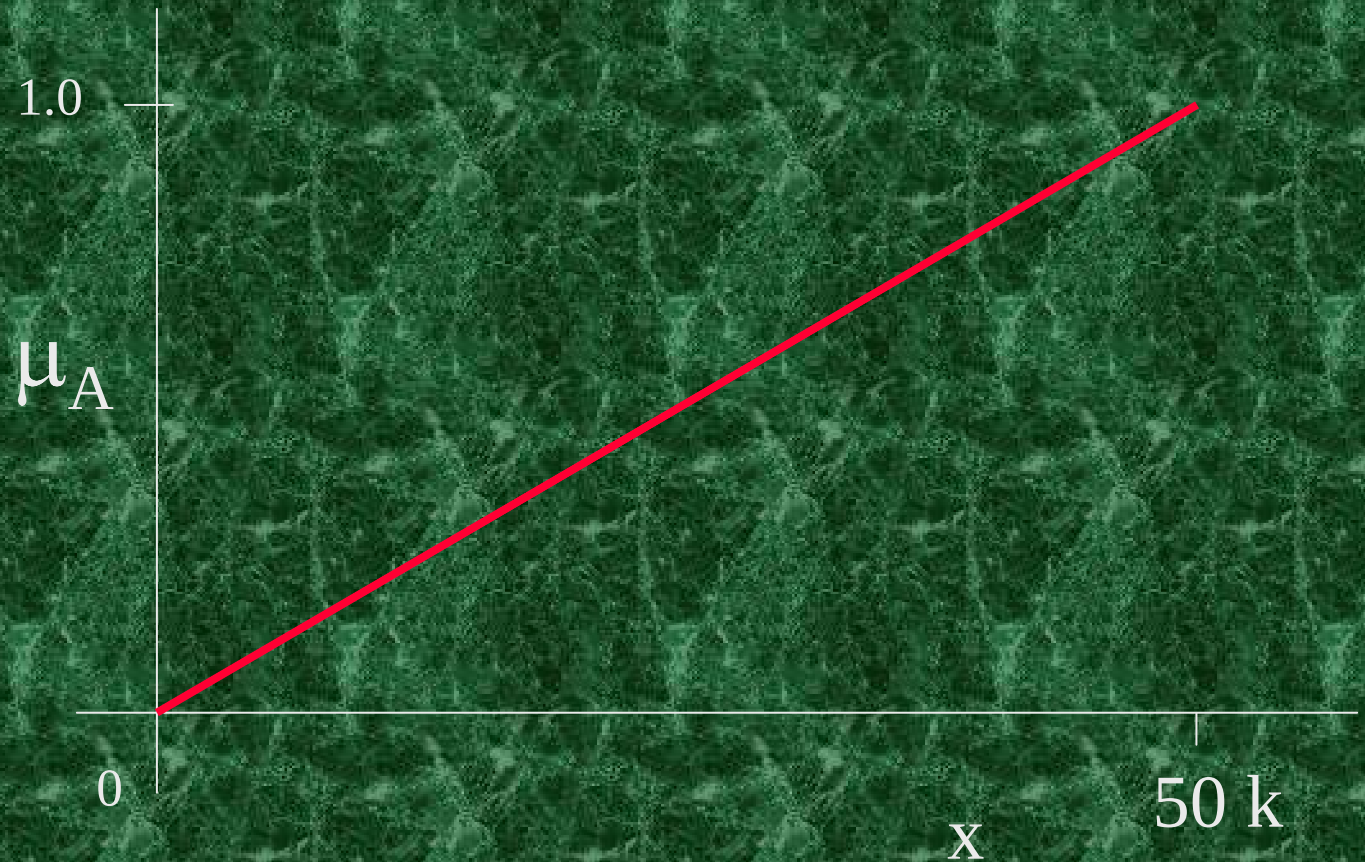
*Peter J. Crickmore, “Multi-site Environmental Liability Analysis -- An Introduction to POMEL Technology”, Calgary, 1993.

Peter J. Crickmore, ”Putting a Dollar Figure on Environmental Risk”, **Calgary Environmental Conference, 1996.

Possibility Example -- POMEL, 2

Complicating the analysis is that different levels of knowledge are possessed about the sites and each site may be subject to different environmental contaminants. For example, a vacant site that may possess some degree of contamination by gasoline (which is volatile) but which has not undergone an intrusive environmental assessment may have its environmental risk represented by the following membership function:

Possibility Example -- POMEL, 3



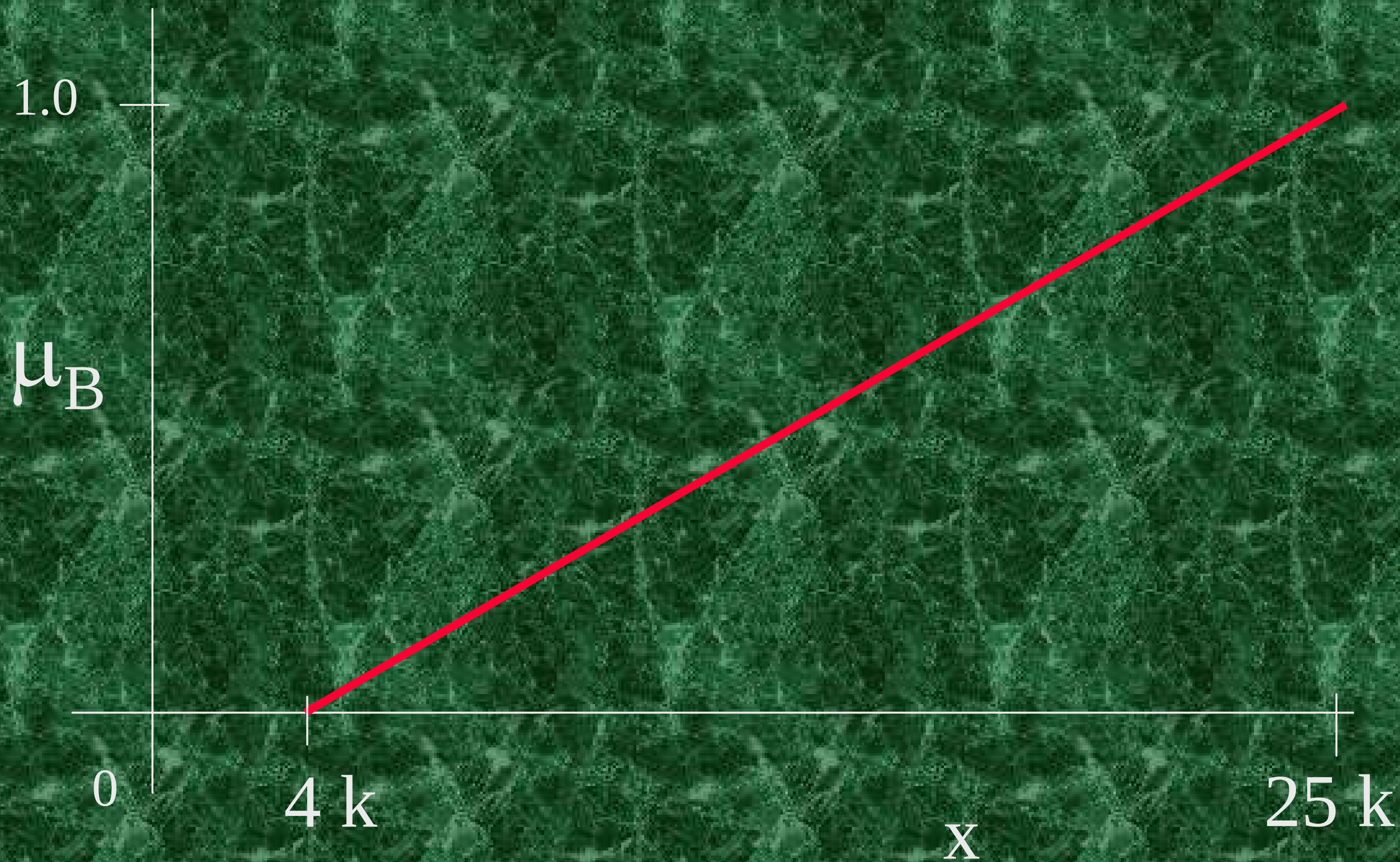
Possibility Example -- POMEL, 4

The membership function should be interpreted as saying that there is some possibility that the site is entirely clean and the cost of remediation is zero and there is some possibility that the site is thoroughly contaminated with the need for $50,000 to be spent on a soil vapour extraction remediation of the site. All other possibilities for contamination extent and remediation costs between these endpoints are equally possible.

Possibility Example -- POMEL, 5

Another site of similar size has had a more thorough environmental assessment and some gasoline contamination on one quarter of the site was found while the other three quarters were clean. The membership function to describe this situation follows:

Possibility Example -- POMEL, 6



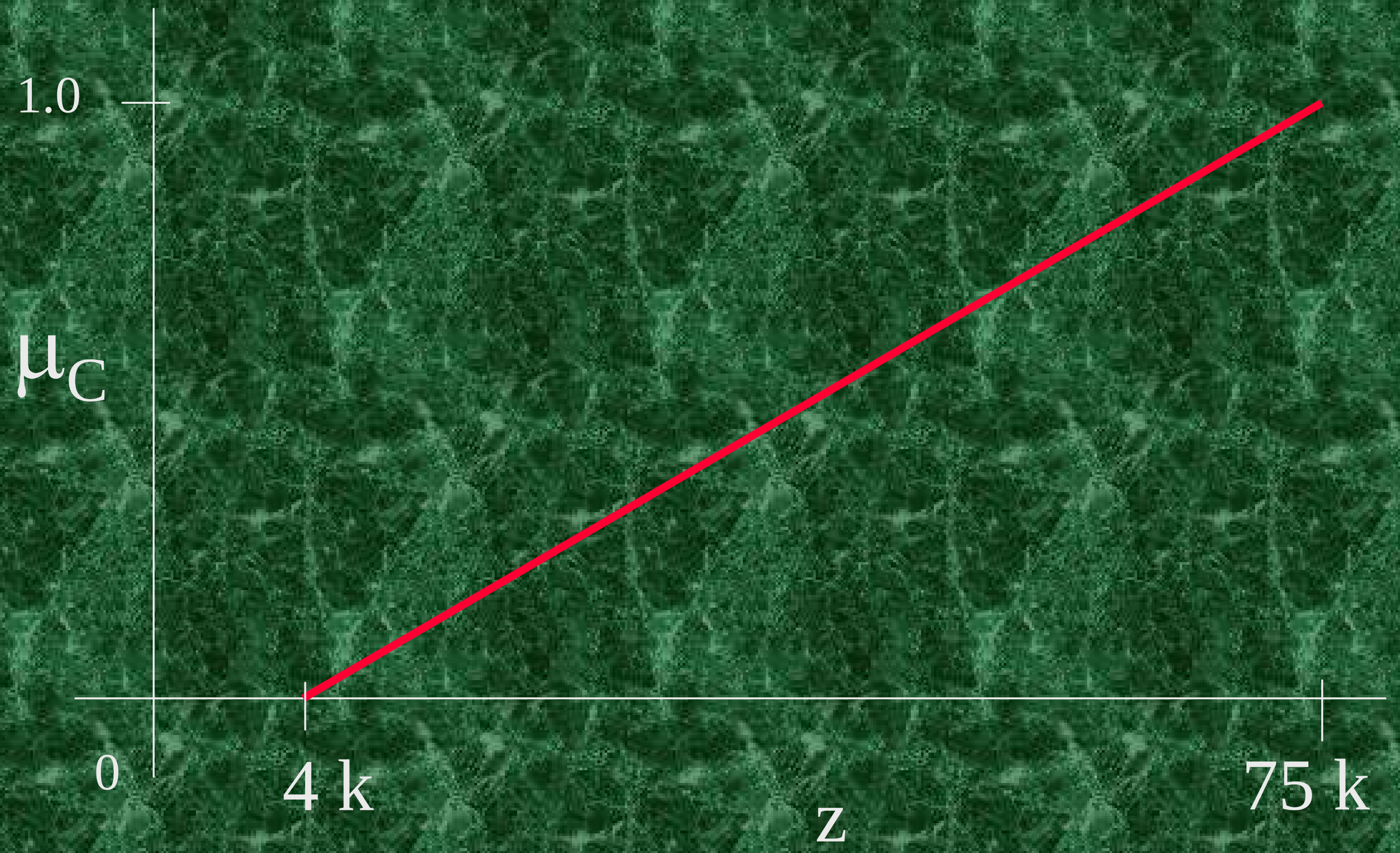
Possibility Example -- POMEL, 7

Note that because there has been some evidence of contamination, there is no longer a possibility of a zero cost for remediation; the $4,000 value at $\mu = 0$ reflects the lowest installed cost of a soil vapour extraction system. Similarly, there is no longer a possibility of a $50,000 remediation cost as three quarters of the site has been determined to be clean; the $25,000 figure represents a scaling of the $50,000 cost to an area of ¼ the size (cost scales as the square root of the area contaminated).

Possibility Example -- POMEL, 8

To consider the environmental risk posed by the “portfolio” of the two sites discussed above, we would simply add the two fuzzy sets, using the addition rules discussed earlier. The membership function of the resulting sum follows:

Possibility Example -- POMEL, 9



Possibility Example -- POMEL, 10

We can then start to use the membership function for our "portfolio" to predict likely cleanup costs (to use for the creation of a contingency fund, for example). If we discard the upper and lower quartiles for our portfolio (equivalent to discarding all z values giving a μ value below 0.25 or above 0.75), we can say that the likely cost will be in the range of \$21,700 to \$57,300.

Possibility Example -- POMEL, 11

A portfolio of a hundred or a thousand sites is built up by adding in sites, one at a time.

Any sites where the information changes (either when a more involved environmental assessment is carried out or when a change in environmental status occurs due to a spill or leak on the site) will change the membership function of the portfolio. This is easily done by subtracting the old membership function of the site from that of the portfolio and then adding the new membership function for the site back in.

Other Fuzzy Applications for Environmental Concerns

Sorting Out Complex Interactions

In a heavily industrialized area, the industries (past and present) and possible contaminants (past and present) form very complex semi-overlapping relationships. To trace a single contaminant back to a source demands tools from fuzzy logic*.

*P.J. Crickmore and J.M. Severyn, "Relationships between Industries and Contaminants: Comparisons between Generalized Inverses and Fuzzy Intersections for Filtering Industry/Contaminant Databases", 75th Canadian Chemical Conference, Edmonton, June 1992.

Other Fuzzy Applications for Environmental Concerns, 2

In an environmental investigation you may have soil analyses, groundwater analyses, soil vapour analyses and sensory observations. Putting these together to describe a contaminant plume is easier with fuzzy logic.

*P.J. Crickmore, B.D. Buoy and W. Goulet, "Interpretation of Piezometer Data: The Use of Fuzzy Sets to Interpret Hydrocarbon Contaminant Plumes", 41st CSChE Conference, Vancouver, October, 1991.

**P.J. Crickmore and J.M. Severyn, "Use of Fuzzy Set Theory to Reconcile Different Measures of Petroleum Contamination", AIChE Spring National Meeting, Houston, March 1993

After applying fuzzy logic, you will feel much less fuzzy.